Bifurcation to large period oscillations in physical systems controlled by delay

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An unusual bifurcation to time-periodic oscillations of a class of delay differential equations is investigated. As we approach the bifurcation point, both the amplitude and the frequency of the oscillations go to zero. The class of delay differential equations is a nonlinear extension of a nonevasive control method and is motivated by a recent study of the foreign exchange rate oscillations. By using asymptotic methods, we determine the bifurcation scaling laws for the amplitude and the period of the oscillations.

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I. INTRODUCTION

Delay problems have been formulated by physicists, engineers, and applied mathematicians for more than a century [1,2]. In its physical interpretation, a constant delay equation describes the effect of a finite interval of the immediate past on the present, and hence on the future. Delays are used to ideally represent the effect of transmission and transportation. Any control system will certainly involve a delay because time is needed to sense the information and then react on it. Delay equations have appeared in various disciplines such as biology [3–7], where the delay accounts for the nervous reaction time, chemistry [8], where a delayed feedback is used to control a chemical reaction, mechanics [9], where machine tool chatter is caused by delay, nonlinear optics [10,11], where the delay comes from unavoidable optical feedback, and car-following models where the driver finite reaction time is taken into account [12–14]. Historically, the study of delay equations started after the First World War principally on account of applied problems that appeared at that time. But the entire subject has grown considerably in recent years due to the development of new mathematical ideas [15,16] and the advances in our computational possibilities [17,18].

A typical phenomenon caused by the delay is the onset of sustained oscillations. These oscillations are undesired in machining processes or dense car traffic. But oscillations generated by a delay are also the basis of important physiological activities such as the respiratory control system [4,19] or the insulin secretory oscillations [20]. An equation describing the growth of a variable y at time t as a function of its value at time $t-\tau$ is called a delay differential equation (DDE). The fixed time interval τ is called the delay or time lag. A linear DDE that appears in many applications is the following first order equation:

$$\frac{dy}{dt} = a[y - y(t-1)], \tag{1}$$

where time is measured in units of τ $(t=t'/\tau)$. Equation (1) typically models a nonevasive control method where the control force vanishes when the target state is reached (here $y=0$) [21]. From an analysis of the characteristic equation, it can be demonstrated that $y=0$ is stable for the interval $0<a<1$ [22]. The control method has been used to stabilize unstable steady states in lasers [23], electronic circuits [24], chemical reactions [25,26], and a magneto-elastic beam system [27]. Equation (1) has also motivated a large number of theoretical studies [28–33].

In this paper, we concentrate on a nonlinear extension of Eq. (1) of the form

$$\frac{dy}{dt} = a[y - y(t-1) - f(y)], \tag{2}$$

where $f(y)=|y|y^n$ $(n=1,3,\ldots)$. The simple case of a quadratic nonlinearity given by

$$\frac{dy}{dt} = a[y - y(t-1) - |y|y] \tag{3}$$

has recently been studied as a model for the oscillations of the exchange rate around its natural value [34]. The first two terms in Eq. (3) describe the growth of the exchange rate by comparing rates at time t and time $t-1$, respectively. If the exchange rate increases because $y(t)>y(t-1)$, it is worthwhile to purchase foreign currency. Hence, the demand for foreign currency goes up and the exchange rate will continue to increase. On the contrary, if the exchange rate decreases because $y(t)<y(t-1)$, the tendency will be to sell foreign currency and the demand will go down. At some time, agents will realize that the absolute deviation $|y(t)|$ increases and they will start to trade. The last term in Eq. (3) describes this effect. Since $dy/dt=-|y|y=-y^2$ if $y>0$ and $dy/dt=-|y|y=y^2$ if $y<0$, $|y|$ will continuously decrease. In practice, the depreciation (or appreciation) of the domestic currency leading to a growth of $|y|$ and the rescuing nonlinear feedback are competing and we intuitively expect an oscillatory regime. The derivation of the original model equation, $dx/dt=a[x-x(t-1)]-b|x|x$, which exhibits two parameters is carefully explained in Ref. [37]. A delay-difference equation similar to the DDE is studied in Ref. [38]. Brunovský *et al.* [35] proved the existence of a periodic solution for all $a>1$. Later, Walther [36] proved that the periodic orbits arise in a bifur-

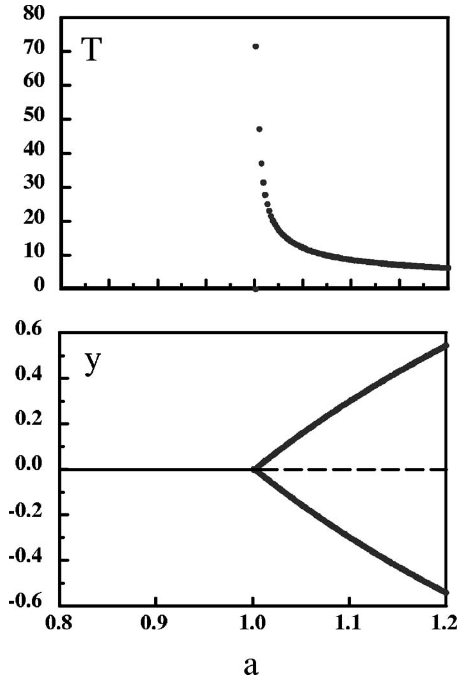

FIG. 1. Numerical bifurcation diagram of the stable oscillations of Eq. (2) with $n=1$ showing the period T (top) and the extrema of y (bottom). Close to $a=1$, the period and the extrema follow an inverse square-root law and a linear law, respectively.

cation from equilibrium at $a=1$, with the minimal period of the oscillations tending to infinity as $a\to 1$. But the way the period and the amplitude of the oscillations change with respect to $a-1$ as a approaches 1 is an open problem and is the main motivation of this paper. Figure 1 shows the numerical bifurcation diagram of the periodic solutions of Eq. (3). The bifurcation at $a=1$ is *not* a Hopf or a homoclinic bifurcation because both the amplitude and the frequency go to zero as we approach the bifurcation point. In this paper, we use singular perturbation techniques [40,41] and construct an asymptotic solution of Eq. (2) valid for a close to 1. We omit all rigorous proofs which would require different tools. The scaling laws for the amplitude and the period of the oscillations strongly differ from the usual bifurcation laws. For Eq. (3), we find that the extrema of oscillations y_M and the period T behave like

$$y_M \sim a-1 \quad \text{and} \quad T \sim (a-1)^{-1/2}, \tag{4}$$

respectively, as $a\to 1^+$.

The plan of the paper is as follows. The scaling of the variables y and t with respect to the small deviation $a-1$ is analyzed in Sec. II. In Sec. III, we construct the leading approximation of the solution of Eq. (2). The bifurcation equation is then investigated in detail and we obtain expressions for the maxima and the period of the oscillations. In Sec. IV, we briefly discuss the mathematical and physical interests of our results.

II. SCALINGS

Numerical simulations of Eq. (3) for values of a close to 1 indicate that the oscillations are nearly harmonic in time and become slower as $a\to 1^+$. This motivates an expansion of $y(t-1)$ as $y(t-1)=y(t)-y'(t)+\frac{1}{2}y''(t)-\frac{1}{6}y'''(t)+\cdots$ where prime means differentiation with respect to time t. Equation (2) then becomes an ordinary differential equation given by

$$y' = a\left[y' - \frac{1}{2}y'' + \frac{1}{6}y''' + \cdots - f(y)\right], \tag{5}$$

where $f(y)=|y|y^n$ $(n=1,3,\ldots)$. The numerical solution also suggests we seek a small amplitude solution. In order to find the right scalings of y,t with respect to the deviation $a-1$, we introduce the new variables u and s

$$y=\varepsilon^p u \quad \text{and} \quad s=\varepsilon^q t, \tag{6}$$

where $p>0$ and $q>0$. The small parameter ε is defined by means of the deviation $a-1$ as

$$a-1=\varepsilon^r c, \tag{7}$$

where $c=\pm 1$ and $r>0$. After expanding $y(t-1)=y(s-\varepsilon^q)$ in Taylor series and inserting Eqs. (6) and (7) into Eq. (2), we find

$$\varepsilon^{p+q+r}cu' + (1+\varepsilon^r c)\Big[-\tfrac{1}{2}\varepsilon^{p+2q}u'' + \tfrac{1}{6}\varepsilon^{p+3q}u''' + \cdots - \varepsilon^{(n+1)p}|u|u^n\Big] = 0, \tag{8}$$

where prime now means differentiation with respect to time s. Equation (8) is the equation of a second order nonlinear oscillator with weak damping if (i) the u'' and $|u|u^n$ terms are the dominant terms as $\varepsilon\to 0$ and (ii) the u' and u''' terms are smaller in magnitude than the leading terms. The first condition is verified if $p+2q=(n+1)p$, or equivalently, if

$$p=2q/n. \tag{9}$$

The second condition is verified if

$$p+q+r=p+3q>p+2q. \tag{10}$$

The equality implies

$$r=2q \tag{11}$$

and the inequality is automatically verified with Eq. (11). Without loss of generalities, we choose $q=1$. Then, $p=2/n$ from Eq. (9) and $r=2$ from Eq. (11).

III. BIFURCATION EQUATION

We now construct an asymptotic solution that follows the scalings laws discussed in the previous section. Specifically, we seek a solution of the form

$$y=\varepsilon^{2/n}(u_0(s)+\varepsilon u_1(s)+\cdots), \tag{12}$$

where $s\equiv\varepsilon t$ and $\varepsilon\equiv\sqrt{(a-1)/c}$ $(c=\pm 1)$. After expanding $y(t-1)=y(s-\varepsilon)$ as

$$y(s-\varepsilon)=y-\varepsilon y' + \tfrac{1}{2}\varepsilon^2 y'' - \tfrac{1}{6}\varepsilon^3 y''' + \cdots, \tag{13}$$

where prime means differentiation with respect to s, we introduce Eq. (12) and $a-1=\varepsilon^2 c$ into Eq. (2) and equate to zero the coefficients of each power of ε. The leading order problem is $O(\varepsilon^{2+2/n})$ and is given by

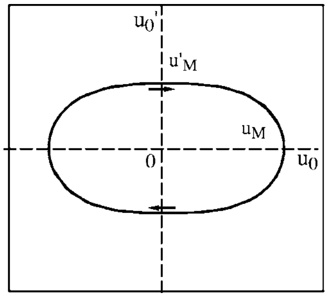

FIG. 2. For each value of the energy $E \geqslant 0$, there is a periodic orbit. Case $n=1$. The maxima of u_0 and u_0' are $u_M=(3E/2)^{1/3}$ and $u_M'=(2E)^{1/2}$, respectively.

$$\tfrac{1}{2}u_0'' + f(u_0) = 0. \tag{14}$$

Equation (14) is conservative and does not depend on the bifurcation parameter. It admits a one parameter family of periodic solutions. The first integral is

$$\frac{1}{2}u_0'^2 + F(u_0) = E, \tag{15}$$

where E is the constant of integration or energy and $F(u_0) \equiv 2\int^{u_0} f(u)du$, i.e.,

$$F(u_0) = \frac{2}{n+2}u_0^{n+2} \quad (u_0 \geqslant 0) \quad \text{and} \quad -\frac{2}{n+2}u_0^{n+2} \quad (u_0 < 0). \tag{16}$$

For every $E \geqslant 0$, there exists a periodic orbit in the phase plane (u_0, u_0'). The maxima of u_0 and u_0' appear when $u_0'=0$ and $u_0=0$, respectively. Using Eq. (16), we then find

$$u_M = \left(\frac{n+2}{2}E\right)^{1/(n+2)} \quad \text{and} \quad u_M' = (2E)^{1/2}. \tag{17}$$

See Fig. 2. Because the amplitude of the oscillations as a function of the bifurcation parameter is still unknown, we need to explore the next problem for u_1.

The problem for u_1 is $O(\varepsilon^{3+2/n})$ and is given by

$$Lu_1 \equiv \frac{1}{2}u_1'' + f'(u_0)u_1 = \tfrac{1}{6}u_0''' + cu_0'. \tag{18}$$

To solve Eq. (18), we note that the linear operator Lu has a one-dimensional null space spanned by u_0'. The operator is self-adjoint, meaning that the solution of the adjoint linear problem is u_0'. Applying the Fredholm alternative, the right-hand side of Eq. (18) needs to satisfy the solvability condition [42]

$$\int_0^P \left(\frac{1}{6}u_0''' + cu_0'\right)u_0' ds = 0 \tag{19}$$

where $P=P(E)$ is the period of $u_0(t)$ corresponding to a value of $E>0$. For algebraic clarity, we denote from now on $u(t)$ as the periodic solution $u_0(t)$ of period P. From Eqs. (14) and (15), $u(t)$ satisfies the equations

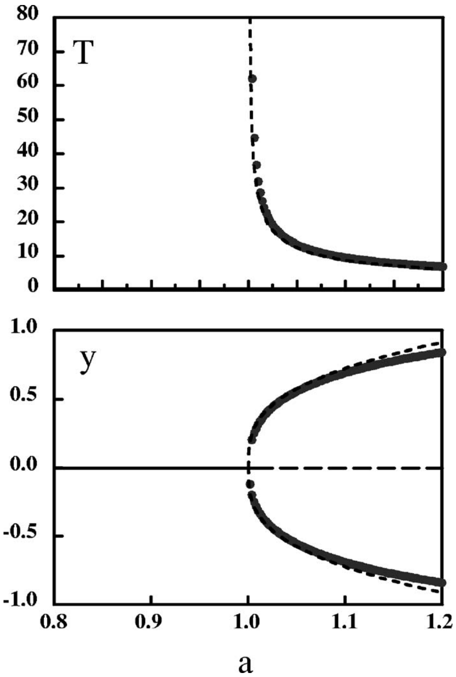

FIG. 3. Comparison between the numerical bifurcation diagram of Eq. (2) with $n=3$ (dots) and the analytical bifurcation diagram given by Eqs. (25) and (28) with $n=3$ (dashed lines). As $a \to 1^+$, the period (top) and the extrema (bottom) change like $(a-1)^{-1/2}$ and $(a-1)^{1/3}$, respectively.

$$u'' = -2u^{1+n} \quad \text{and} \quad u' = \sqrt{2\left(E - \frac{2}{n+2}u^{n+2}\right)} \tag{20}$$

for $u \geqslant 0$ and $u' \geqslant 0$. We determine u''' as $u'''=-2(n+1)u^n u'$ and introduce a change of variable from s to u using $ds = du/\sqrt{2[E-[2/(n+2)]u^{n+2}]}$ from the expression of u' in Eq. (20). After simplifying, the integral (19) becomes

$$-\frac{n+1}{3}\int_0^{u_M} u^n \sqrt{E - \frac{2}{n+2}u^{n+2}}\, du + c\int_0^{u_M} \sqrt{E - \frac{2}{n+2}u^{n+2}}\, du = 0. \tag{21}$$

Introducing $u=u_M v$, Eq. (21) can be simplified as

$$-\frac{n+1}{3}u_M^n I_1 + cI_2 = 0, \tag{22}$$

where I_1 and I_2 are the following two definite integrals:

$$I_1 \equiv \int_0^1 v^n \sqrt{1-v^{n+2}}\, dv \quad \text{and} \quad I_2 \equiv \int_0^1 \sqrt{1-v^{n+2}}\, dv \tag{23}$$

which *contain no parameters*. Therefore, we may extract u_M^n from (22) and obtain u_M as

$$u_M = \left(\frac{3I_2}{(n+1)I_1}c\right)^{1/n}. \tag{24}$$

In terms of the original variable, the maximum of the oscillations is given by $y_M=\varepsilon^{2/n}u_M$ and using Eq. (24), we obtain

$$y_M=\left(\frac{3I_2}{(n+1)I_1}(a-1)\right)^{1/n}. \quad (25)$$

We next concentrate on the period. Using Eq. (15), we determine the period in s as

$$P=\frac{4}{\sqrt{2}}\int_0^{u_M}\frac{1}{\sqrt{E-2u_0^{n+2}/(n+2)}}\,du_0=\frac{2\sqrt{n+2}}{u_M^{n/2}}I_3 \quad (26)$$

where

$$I_3\equiv\int_0^1\frac{1}{\sqrt{1-v^{n+2}}}\,dv. \quad (27)$$

The expression (26) then implies a period in time t given by

$$T=2\sqrt{n+2}I_3\frac{1}{y_M^{n/2}}. \quad (28)$$

Figure 3 illustrates the case $n=3$. The period follows an inverse square-root law as for the case $n=1$ and the extrema follows a cubic-root law. The broken line corresponds to the asymptotic approximations given by Eqs. (25) and (28) where the integrals have been computed numerically.

IV. DISCUSSION

We examined a class of DDEs described by Eq. (2) with $f(y)=|y|y^n$ and showed that there exists a bifurcation to long period oscillations exhibiting the scaling laws

$$y_M\sim(a-1)^{1/n} \quad \text{and} \quad T\sim(a-1)^{-1/2} \quad (29)$$

as $a\to1^+$. As the nonlinearity becomes stronger (n becomes larger), the amplitude of the oscillations exhibits a progressively larger scaling law, but the period keeps the same scaling law whatever n. These unusual properties are not related to the absolute value $|y|$ in $f(y)$. We have analyzed the case $f(y)=y^{2+n}$ ($n=1,3,\ldots$) and obtained similar results. Mathematically, the bifurcation point $a=1$ corresponds to a double zero eigenvalue and can be analyzed by center manifold techniques [39,43]. We have verified that the case $n=1$ can be reduced to Eq. (8.38) in Ref. [43], p. 318 with $g_{00}(\alpha)=g_{10}(\alpha)=g_{02}(\alpha)=0$. However, we cannot make the successive change of coordinates described in Ref. [43] because the nonlinear function involves an absolute value. To investigate our bifurcation problem, we preferred to use expansion techniques and apply solvability conditions.

From a physical point of view, this bifurcation is interesting because it suggests a new mechanism for the generation of low-frequency oscillations. Although, we may not avoid the instability of the basic solution $y=0$ as soon as a surpasses 1, the bifurcation is as best we may wish in the context of control. Slightly above $a=1$, the oscillations are slow and of small amplitude so that the physical impact of the instability remains limited.

ACKNOWLEDGMENTS

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